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UNIQUE ERGODICITY OF THE HOROCYCLE FLOW: VARIABLE NEGATIVE CURVATURE CASE

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ABSTRACT
H. Furstenberg showed that horocycle flows on compact manifolds of constant negative curvature are uniquely ergodic. This paper generalizes his result to the case of variable negative curvature, in the more general context of flows whose orbits are the unstable manifolds of certain Anosov flows.

0. Introduction

Horocycles were introduced to dynamical systems in the study of the dynamics of geodesic flows. For a compact connected orientable 2-manifold N of negative curvature, a horocyle is an unstable manifold of the geodesic flow (i.e., the set of all points, in the unit tangent bundle (T_1N) of N, which are backwards asymptotic with a given point under the action of the geodesic flow). Classically, horocycles were defined geometrically as certain curves in the universal covering manifold $\tilde{N}$ of N ([6]). Our horocycles are related to the classical horocycles as follows: take one of our horocycles, project it into N, and then lift that to a curve in $\tilde{N}$; the result being a classical horocycle. (See [1, p. 30], [2].)

A continuous 1-parameter flow (on T_1N) whose orbits are the horocycles is called a horocycle flow. We will use the geodesic flow to study the dynamics of horocycle flows. In particular, we show ((3.6)) that horocycle flows are uniquely ergodic (i.e., they have a unique invariant Borel probability measure), provided the geodesic flow is of class C^2 (the C^2 assumption is apparently unnecessary; see (1.8)). This was proved in the case of constant negative curvature by Harry Furstenberg ([5]). His paper provided the motivation for this work, although our method is different.

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What we actually prove ((3.5)), which was suggested by Charles Pugh, is more general than (3.6), namely: continuous flows (called W^u flows), whose orbits are the unstable manifolds of suitable (see (1.7) and (1.8)) Anosov flows, are uniquely ergodic. (The geodesic flow of manifolds of negative curvature is the classical example of an Anosov flow.) The crucial (but mild) assumption is that the W^u flow be minimal. The same thing works just as well for flows whose orbits are stable manifolds. With our parametrization (2.1) of the W^u flow the unique invariant measure is the one which maximizes entropy for the Anosov flow; this measure is constructed in [12].

We solved the analogous problem for Anosov diffeomorphisms in the more general context of Axiom A attractors ([11]). The results of this paper will be extended to the Axiom A context in a joint paper with Rufus Bowen ([4]), using another method.

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1. Anosov background

Let M be a compact connected Riemannian manifold and $\{f_t\}$ an Anosov flow. This means that $\{f_t\}$ is a (C^r-) differentiable flow without fixed points and there is a (Df_t)-invariant continuous splitting of the tangent bundle $TM = E^u \oplus E^s \oplus E$, where E is the line bundle tangent to the flow direction and E^u and E^s satisfy:

(1.0) there exist constants $a > 0$, $0 < \mu < 1$ such that for $t \geqq 0$: if $v \in E^u$, then $\|Df_{-t}(v)\| \leqq a\mu^t \|v\|$, and if $v \in E^s$, then $\|Df_t(v)\| \leqq a\mu^t \|v\|$. Let $l = \dim E^u$, $k = \dim E^s$.

We will need some facts from stable manifold theory. Let d be the induced Riemannian metric on M. The unstable manifold

$$(W^u(x)) = \{y \in M : \lim_{t \to +\infty} d(f_{-t}x, f_{-t}y) = 0\}.$$

The stable manifold

$$(W^s(x)) = \{y \in M : \lim_{t \to +\infty} d(f_t x, f_t y) = 0\}.$$

The weak stable manifold

$$(W^{ws}(x)) = \bigcup_{t \in \mathbf{R}} W^s(f_t x).$$

By a *continuous family of submanifolds*, we mean a partition of M into injectively immersed q-dimensional submanifolds $\{W(x)\}$ such that for each $x \in M$, there is a neighborhood U of x and a continuous map $g: U \to C^r(D^q, M)$ such that $g(y)$ embeds D^q in $W(y)$ and $g(y)(0) = y$ for $y \in U$.

LEMMA (1.1) ([1], [13]). *$\{W^u(x)\}$, $\{W^{ws}(x)\}$ form continuous families of immersed submanifolds, whose tangent fields are $E^u, E^s \oplus E$. Actually, each $W^u(x)$ is an immersed copy of $\mathbf{R}^l$.*

Let $J_x: \mathbf{R}^l \to W^u(x)$ be the immersion. This gives $W^u(x)$ a topology (the *intrinsic* topology). Now, we define the local manifolds. The local unstable manifold

$$(B^u_\varepsilon(x)) = \{y \in W^u(x): d^u_x(x, y) \leqq \varepsilon\}$$

and the local weak stable manifold

$$(B^{ws}_\varepsilon(x)) = \{y \in W^{ws}(x): d^{ws}_x(x, y) \leqq \varepsilon\},$$

where d^u_x, d^{ws}_x are the induced Riemannian metrics on $W^u(x)$, $W^{ws}(x)$. A transversality argument gives:

LEMMA (1.2) (Canonical Coordinates, [1], [13]). *For sufficiently small $\eta > 0$, there exists $\gamma = \gamma(\eta)$, $0 < \gamma < \eta$, such that if $d(y, z) \leqq 2\gamma$, then $[y, z] \equiv B^{ws}_\eta(y) \cap B^u_\eta(z)$ is a single point; $[\cdot, \cdot]$ is continuous on $\{(y, z) \in M \times M: d(y, z) \leqq 2\gamma\}$.*

NOTATION. Let $[A, B] = \{[a, b]: a \in A, b \in B\}$.

NOTE. The topology that $B^u_\eta(x)$ ($B^{ws}_\eta(x)$) inherits from $W^u(x)$ ($W^{ws}(x)$) is the same as the topology it inherits from M.

LEMMA (1.3). (a) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then there is a homeomorphism $h: B^u_\gamma(y) \times I \to M$ such that for each $z \in B^u_\gamma(y)$, $h(z, \cdot)$ is a smooth curve in $W^{ws}(z)$ of length $\leqq \varepsilon$ and $h(z, 1) = z$, $h(z, 0) = [z, x]$.*

(b) *Given $\varepsilon' > 0$ there exists $\delta' > 0$ such that if $d(x, y) < \delta'$, then for all $t \geqq 0$ and $z \in B^u_\gamma(y)$, $d(f_t z, f_t[z, x]) \leqq \varepsilon'$.*

PROOF. (a) By (1.1) (the continuity of $\{W^{ws}\}$), there is a homeomorphism $g_x: B^u_\eta(x) \times D^{k+1} \to M$ onto a neighborhood $N(x)$ of x such that $g_x(w, 0) = w$, each $g_x(w, \cdot)$ is a smooth embedding into $W^{ws}(w)$ and the map $B^u_\eta(x) \to$

$C^r(D^{k+1}, M)$ defined by $w \to g_x(w, \cdot)$ is continuous. (We are assuming here that the η chosen (in (1.2)) is suitably small, independent of x; this can be done by compactness of M.)

Now if $z \in N(x)$ and $d(z,x) < 2\gamma$, define $p_z \in D^{k+1}$ by $g_x^{-1}(z) = ([z,x], p_z)$. It follows by (1.1) (the continuity of W^u) that for small $\delta > 0$, if $d(x,y) < \delta$, then $B_\gamma^u(y) \subset N(x)$ and the map

$$h: B_\gamma^u(y) \times I \to M, \quad h(z,t) = g_x([z,x], tp_z)$$

will have the required properties. Moreover, by compactness of M, δ can be chosen independent of x.

(b) By (a) we get that if $d(x,y) < \delta$ and $z \in B_\gamma^u(y)$, then $[z,x] \in B_\varepsilon^{ws}(z)$. By the contractiveness of Df_t on E^s (given in (1.0)), it follows that there is a constant β independent of ε such that for all $z \in M$ and $t \geqq 0$

$$f_t B_\varepsilon^{ws}(z) \subset B_{\varepsilon\beta}^{ws}(f_t z) \subset B_{\varepsilon\beta}(f_t z),$$

the ball of radius $\varepsilon\beta$ in the metric d. This gives the lemma. ■

Definition (1.4). A W^u flow is a continuous 1-parameter flow whose orbits are the unstable manifolds, $W^u(x)$.

A necessary and sufficient condition that the W^u-family (i.e., the family of $W^u(x)$'s) admits a W^u flow is that E^u be orientable and 1-dimensional. So, we will need to assume this. Now, pick an orientation of E^u. This amounts to a continuous vector field $V: M \to TM$, which spans E^u. V coherently and continuously orients each curve, $W^u(x)$: we can choose the immersion $J_x: \mathbf{R} \to W^u(x)$ such that $(\dot{J}_x(0)) \cdot (V(x)) > 0$. Define $W_+^u(x) = J_x([0, +\infty))$ and $W_-^u(x) = J_x((-\infty, 0])$; this simply gives a notion of "right and left". If $x \in A \subset W^u(x)$, $g: A \to W^u(gx)$ is 1–1 and $g(A \cap W_\pm^u(x)) \subset W_\pm^u(gx)$, then we say that g is orientation preserving.

Lemma (1.5). *If $y \in B_\gamma^{ws}(x)$, then $P_{y,x}: B_\gamma^u(y) \to B_\eta^u(x)$, defined by $P_{y,x}(z) = [z,x]$, is orientation preserving.*

Proof. Let $G = \{y \in B_\gamma^{ws}(x): P_{y,x}$ is orientation preserving$\}$. By continuity of $[\cdot,\cdot]$, G is open and closed in $B_\gamma^{ws}(x)$. G is non-empty since $P_{x,x}$ is the identity. Since $B_\gamma^{ws}(x)$ is connected (it's a disk), G must be all of $B_\gamma^{ws}(x)$. ■

Lemma (1.6). *For each t and x, f_t is orientation preserving on $W^u(x)$.*

PROOF. First note that $f_t W^u(x) = W^u(f_t x)$. Now, $Df_t(V(x)) = b(x,t)\cdot V(x)$ for some continuous non-zero function b. Since $b(x,0)=1$, $b(x,t)$ is always positive, whence the lemma. ■

For the remainder of the paper, we assume:

(1.7)

(a) E^u is orientable and 1-dimensional.

(b) $\{f_t\}$ is C^2.

(c) $\{f_t\}$ preserves a smooth measure.

(d) Each $W^u(x)$ is dense.

REMARKS (1.8). We assume (b) and (c), because they are assumed in the reference we use to get a system of uniformly expanding measures (see (2.1)); however, one can produce these measures by using symbolic dynamics, without these assumptions (this will essentially appear in [4]). Also, (d) is equivalent to each of two other assumptions:

(d)′ $\{f_t\}$ is *not* the constant time suspension of an Anosov diffeomorphism.

(d)″ $\{f_t\}$ has no continuous eigenfunctions (see [1, theors. 13, 14, 15]).

Also, it is easy to see that (d)′ is necessary for our result.

2. Construction of a convenient parametrization

A W^u family may admit several different W^u flows, but of course they all have the same orbits. Hence, if one is uniquely ergodic, so is any other ([10]). We will take advantage of this and construct a convenient W^u flow $\{\phi_s\}$ which satisfies two useful "commutation relations", showing how it "commutes" with $\{f_t\}$ and $[\cdot,\cdot]$.

NOTATION. $\phi_{(a,b)}(x) = \{\phi_s(x) \colon s \in (a,b)\}$.

PROPOSITION (2.1). *The W^u family admits a W^u flow $\{\phi_s\}$ such that*

(CRI) *for some constant $\lambda > 1$, $f_t \circ \phi_s = \phi_{\lambda^t s} \circ f_t$ for all s and t (i.e., the Anosov flow uniformly expands the orbits of the W^u flow).*

(CRII) *For all y in M, $\phi_{(-1,1)}(y) \subset B^u_\gamma(y)$ and if $d(x,y) \leqq \gamma$, then there is a strictly increasing Lipschitz function $k_{x,y} \colon (-1,1) \to \mathbf{R}$, defined by $\phi_{k_{x,y}(s)}(x) = [\phi_s(y), x]$, such that for each x*

(a) $\lim_{y\to x} k_{x,y}(s) = s$, *uniformly in s.*

(b) $\lim_{y\to x} (d/ds)\, k_{x,y}(s) = 1$, *uniformly in s.*

NOTE. $(d/ds)k_{x,y}(s)$ exists a.e. $s \in (-1,1)$, since $k_{x,y}$ is Lipschitz; at the points of ambiguity define $(d/ds)k_{x,y}(s) = 1$. Also, $(d/ds)k_{x,y}(s)$ will be the Jacobian of

$$P_{y,x} : (B^u_\gamma(y), \mu_y) \to (B^u_\eta(x), \mu_x), \qquad z \to [z, x]$$

(μ_x, μ_y below).

PROOF. In [12], assuming (1.7)b, c, (1.8)d″, Margulis constructs a nice system of measures $\{\mu_x\}$ (each μ_x defined on $W^u(x)$); we will use these to construct $\{\phi_s\}$. Specifically, the $\{\mu_x\}$ satisfy: (they are defined on the Borel field of $W^u(x)$).

(0) If $y \in W^u(x)$, then $\mu_y = \mu_x$.

(1) There is a constant $\lambda > 1$ such that for all $t \in \mathbf{R}$, $x \in M$ and A in $W^u(x)$, $\mu_{f_t x}(f_t A) = \lambda^t \mu_x(A)$.

(2) If U is a compact neighborhood in $W^u(x)$, then $0 < \mu_x(U) < +\infty$.

(3) For $\varepsilon > 0$, if x and y are sufficiently close and $A \subset B^u_\gamma(y)$, then

$$|\mu_y(A) - \mu_x([A, x])| < \varepsilon \cdot \min(\mu_y(A), \mu_x([A, x])).$$

(4) μ_x is zero on points.

(5) $\mu_x(W^u_\pm(x)) = +\infty$.

NOTE. (0), (1) and (2) are essentially stated in [12, p. 64 (bottom)]; (3) follows from (1.3a) and [12, Lemma 3.12, p. 65]. We now deduce (4) and (5) from (1), (2), and (3):

PROOF OF (4). If not, then by (1) there exist points of arbitrarily large measure. So, by compactness of M, there is a convergent sequence $x_n \to x$ such that $\mu_{x_n}(\{x_n\}) \to +\infty$. Since $d(x_n, [x_n, x]) \to 0$, it follows from (3) that for sufficiently large n, $\mu_x(\{[x_n, x]\}) > \frac{1}{2}\mu_{x_n}(\{x_n\})$. So, $\mu_x(B^u_\eta(x)) = +\infty$, contrary to (2). ■

PROOF OF (5). First note

$$\inf_{x \in M} \mu_x(B^u_\eta(x)) > 0; \tag{6}$$

for if x and y are sufficiently close, then by (3)

$$\mu_x(B^u_\eta(x)) \geqq \mu_x([B^u_\gamma(y), x]) > \tfrac{1}{2}\mu_y(B^u_\gamma(y)).$$

Now, use compactness of M, to get (6).

For each x, there is an $x' \in W^u(x)$ such that $B^u_\eta(x') \subset W^u_+(x)$; this implies that $W^u_+(x)$ contains infinitely many disjoint $B^u_\eta(x')$'s. This and (6) yield $\mu_x(W^u_+(x)) = +\infty$; similarly $\mu_x(W^u_-(x)) = +\infty$.

Definition of $\{\phi_s\}$. For $s \geqq 0$, let $\phi_s(x)$ be the point in $W^u_+(x)$ such that the μ_x-measure of the arc from x to $\phi_s(x)$ in $W^u(x)$ is s. Similarly for $s \leqq 0$ (replace $W^u_+(x)$ by $W^u_-(x)$). That this is well-defined follows from (2), (4), and (5). By (2), the ϕ-orbit of x is all of $W^u(x)$, and the group property $\phi_{t+s} = \phi_t \circ \phi_s$ is immediate. Except for continuity (which will follow from CRII), this shows that $\{\phi_s\}$ is a W^u flow.

CRI follows from (1) and (1.6). For CRII, first note that there exists $\alpha > 0$ such that for all y in M, $\phi_{(-\alpha,\alpha)}(y) \subset B^u_\gamma(y)$; this follows from (6) above, for (6) is valid for η replaced by any smaller number, e.g., $\gamma/2$. Then by a uniform dilation of the parametrization, we may assume that $\alpha = 1$; this affects nothing we have done so far. Each $k_{x,y}$, as defined in CRII, is strictly increasing by (1.5). By (3), given $\varepsilon > 0$, if y is sufficiently close to x, then

$$\left| \frac{k_{x,y}(t) - k_{x,y}(s)}{t - s} - 1 \right| < \varepsilon$$

for all $s, t, -1 \leqq s < t \leqq 1$. This shows that $k_{x,y}$ is Lipschitz and gives us CRIIb. It will also yield CRIIa once we know that $\lim_{y \to x} k_{x,y}(0) = 0$. Now to see the latter, first note that for fixed x the map $s \to \phi_s(x)$ is 1–1 and continuous (by (2) and (4)); so for some β_1, β_2, the map $[\beta_1, \beta_2] \to B^u_\eta(x)$ $(s \to \phi_s(x))$ is a homeomorphism onto $B^u_\eta(x)$. Now since $\phi_{k_{x,y}(0)}(x) \in B^u_\eta(x)$, and

$$\lim_{y \to x} \phi_{k_{x,y}(0)}(x) = \lim_{y \to x} [y, x] = [x, x] = x,$$

we must have $\lim_{y \to x} k_{x,y}(0) = 0$. So, this gives CRII.

Now for continuity (of the map ϕ: $(x, s) \to \phi_s(x)$). First note that by CRIIa, if $x \in M$ and $|t| < 1$, $k_{x,y}(s)$ is continuous in (y, s) at (x, t); for

$$|k_{x,y}(s) - k_{x,x}(t)| = |k_{x,y}(s) - t| \leqq |k_{x,y}(s) - s| + |s - t|.$$

Also, $\phi_s(y) = [\phi_{k_{x,y}(s)}(x), y]$. This fact plus continuity of the three maps: $k_{x,y}(s)$ (in (y, s)), $s \to \phi_s(x)$, (for x fixed), and $[\cdot, \cdot]$ yields continuity of ϕ at (x, t). So, $\phi|_{M \times (-1,1)}$ is continuous. Now use the group property to get continuity of ϕ everywhere.

3. Proof of main result

For $n \in Z^+$, choose t_n such that $\lambda^{t_n} = 2^n$ (λ as in (2.1)). For $h \in C(M)$, define $R_n h$ by

$$R_n h(x) \equiv \frac{1}{2^n} \int_0^{2^n} h \circ \phi_s \circ f_{t_n}(x)\, ds = \int_0^1 h \circ f_{t_n} \circ \phi_s(x)\, ds$$

(the last equality follows from CRI).

PROPOSITION (3.1). *$\{R_n h\}$ converges uniformly to a constant as $n \to +\infty$.*

We will use (3.1) to show that "time averages" of continuous functions, h, computed over orbits of $\{\phi_s\}$, converge to a constant (independent of starting point), and this implies unique ergodicity.

The proof of (3.1) requires a few lemmas. The basic idea is that $\{R_n h\}$ averages h over larger and larger (as $n \to +\infty$) subarcs of unstable manifolds; we will then get (3.1) from

(III): the orbits of $\{\phi_s\}$ are dense, and

(IV): for each $x \in M$,

$$\{f_t \mid_{B_\eta^{ws}(x)}\}_{t \geqq 0}$$

is equicontinuous (uniformly in x) (see (1.3b)).

In fact, our proof essentially shows that any flow $\{\phi_s\}$, which satisfies CRI, CRII, III, and IV, with respect to another flow $\{f_t\}$ and local sections (to $\{\phi_s\}$) $B_x \equiv B_\eta^{ws}(x)$, is uniquely ergodic.

LEMMA (3.2). *For each h, $\{R_n h\}_{n \geqq 0}$ is an equicontinuous family of uniformly bounded functions.*

PROOF. It follows from (1.3b) that, given x and $\varepsilon > 0$, if y is sufficiently close to x, then

$$|h \circ f_t(\phi_s(y)) - h \circ f_t([\phi_s(y), x])| < \varepsilon \quad \text{for } |s| < 1 \text{ and } t \geqq 0.$$

By CRII, we may also assume $|k_{x,y}(s) - s| < \varepsilon$ for $|s| < 1$ and $|(d/ds)\, k_{x,y}(s) - 1| < \varepsilon$ a.e. $|s| < 1$. We claim that for all $n \geqq 0$

$$|R_n h(x) - R_n h(y)| < \varepsilon \cdot (1 + 3\|h\|). \tag{3.2.1}$$

To see this, first note that, by choice of x, y, and ε,

$$(3.2.2) \qquad \left| R_n h(y) - \int_0^1 h \circ f_{t_n}([\phi_s(y), x])\, ds \right|$$
$$= \left| \int_0^1 (h \circ f_{t_n} \circ \phi_s(y) - h \circ f_{t_n}([\phi_s(y), x]))\, ds \right| < \varepsilon.$$

By definition of $k_{x,y}$ (as in CRII),

$$(3.2.3) \qquad \int_0^1 h \circ f_{t_n}([\phi_s(y), x])\, ds = \int_0^1 h \circ f_{t_n} \circ \phi_{k_{x,y}(s)}(x)\, ds.$$

Now, since $|(d/ds)\, k_{x,y}(s) - 1| < \varepsilon$,

$$(3.2.4) \qquad \left| \int_0^1 h \circ f_{t_n} \circ \phi_{k_{x,y}(s)}(x)\, ds - \right.$$
$$\left. \int_0^1 \left(\frac{d}{ds} k_{x,y}(s)\right) \cdot (h \circ f_{t_n} \circ \phi_{k_{x,y}(s)}(x))\, ds \right| < \varepsilon \|h\|.$$

Since $k_{x,y}$ is Lipschitz, hence absolutely continuous, we can use the change of variables formula:

$$(3.2.5) \qquad \int_0^1 \left(\frac{d}{ds} k_{x,y}(s)\right) \cdot (h \circ f_{t_n} \circ \phi_{k_{x,y}(s)}(x))\, ds = \int_{k_{x,y}(0)}^{k_{x,y}(1)} h \circ f_{t_n} \circ \phi_s(x)\, ds.$$

And

$$(3.2.6) \qquad \left| \int_{k_{x,y}(0)}^{k_{x,y}(1)} h \circ f_{t_n} \circ \phi_s(x) ds - R_n h(x) \right| < \|h\| \cdot |k_{x,y}(1) - 1 + k_{x,y}(0)| < 2\varepsilon \|h\|.$$

Now, (3.2.2) − (3.2.6) yields (3.2.1), and hence (3.2). ■

LEMMA (3.3). *Given $\varepsilon^* > 0$ there exists an integer $s^* > 0$ such that for all $y \in M$, $\{\phi_j(y) : j = 0, 1, \cdots, s^*\}$ is ε^* dense.*

PROOF. This will follow immediately from the compactness of M, once we know that the map ϕ_1 is minimal. But actually for each $t \neq 0$, the map ϕ_t is minimal. If not, then there would be a proper minimal set for some ϕ_t, whence by the minimality of the flow $\{\phi_t\}$ (1.7d) there would be a continuous eigenfunction for the flow; i.e., a function f satisfying $f(\phi_s(x)) = [\exp 2\pi i s/t] f(x)$, $|f| = 1$. Joe Plante pointed out that this would give rise to a non-trivial asymptotic cycle ([2, pp. 147–152]), contrary to [14, Theorem 2.4], modulo some smoothness technicalities.

Alternatively, one could get the minimality of ϕ_1 from more general considerations: as several people have pointed out, if a continuous flow on a separable metric space is minimal, then all but countably many of the time t maps are also minimal; otherwise, there would be uncountably many different eigenfunctions as above, contradicting the separability of $C(M)$. Now, we can assume that ϕ_1 is minimal by reparametrizing the flow by an appropriate uniform dilation of the orbits.

By a straightforward computation, one gets

LEMMA (3.4). *For all* $m \geqq 0$,

$$R_{n+m}h = \frac{1}{2^m}\sum_{j=0}^{2^m-1} R_n h \circ \phi_j \circ f_{t_m}.$$

PROOF OF (3.1). Set $c_n = \min_{x\in M} R_n h(x)$. By Lemma (3.4), $\{c_n\}$ is a non-decreasing sequence. Set $c = \lim c_n$. Now let $\{n_k\}$ be a sequence with $R_{n_k}h \to g$ uniformly. Note that g is a continuous function with minimum c. By Lemma (3.4),

$$R_{n_k+m}h \to \frac{1}{2^m}\sum_{j=0}^{2^m-1} g \circ \phi_j \circ f_{t_m} \equiv \bar{g}.$$

$\bar{g}$ also has minimum c. Now if $\bar{g}(x_0) = c$, then $g(\phi_j \circ f_{t_m}(x_0)) = c$ for $0 \leqq j \leqq 2^m - 1$. Thus by (3.3) given ε^*, g takes the value c on an ε^* dense set, since m above is arbitrary. But then $g \equiv c$. By (3.2) and the Arzela–Ascoli theorem, $R_n h \to c$ uniformly.

Now we prove

THEOREM (3.5). *Every W^u flow, whose corresponding Anosov flow satisfies* (1.7), *is uniquely ergodic.*

PROOF. As mentioned before, it suffices to show that the specific parametrization $\{\phi_s\}$ is uniquely ergodic. Let $h \in C(M)$ and $\varepsilon > 0$. By (3.1), there exists n such that for all $y \in M$, $|R_n h(y) - c| < \varepsilon$. Now, for each $j \in Z^+$ and $x \in M$,

$$\frac{1}{2^n j}\int_0^{2^n j} h \circ \phi_s(x)\, ds \quad \text{is an average of } \{R_n h(f_{-t_n} \circ \phi_{2^n i}(x))\}_{i=0}^{j-1}.$$

Thus, for all $j \in Z^+$,

$$\left| \frac{1}{2^n j}\int_0^{2^n j} h \circ \phi_s(x)\, ds - c \right| < \varepsilon.$$

Since $\{2^n j\}_{j \in Z^+}$ increases to $+\infty$ with bounded increments, this implies that, for t sufficiently large,

$$\left| \frac{1}{t} \int_0^t h \circ \phi_s(x)\, ds - c \right| < 2\varepsilon.$$

So, for all $x \in M$,

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t h \circ \phi_s(x)\, ds = c.$$

This is precisely the time average criterion for unique ergodicity ([9]). ∎

Now, if N is a compact connected orientable 2-manifold of negative curvature, then its geodesic flow (on $M = T_1N$, the unit tangent bundle) satisfies (1.7)a:

First orient N. For $(p, v) \in T_1N$ (i.e., $p \in N$, $v \in (TN)_p$), orient $W^u((p, v))$ as follows: letting π: $T_1N \to N$ denote the projection map, $\pi W^u((p, v))$ is a curve in N passing through p; it is also orthogonal to the geodesic which passes through p in direction v (see [2]); now choose a unit vector w, tangent to $\pi W^u(p, v)$ at p such that the ordered frame $\{v, w\}$ agrees with the orientation of N at p. Now, lift w to a tangent vector to $W^u((p, v))$ at (p, v); this defines an orientation of E^u (i.e., a continuous vector field which spans E^u).

The geodesic flow also satisfies 1.7c ([3]) and 1.8d″ ([3, p. 101]). Recalling that a horocycle flow is a W^u flow whose corresponding Anosov flow is the geodesic flow, we get from (3.5):

Main Result (3.6). *Every horocycle flow, whose corresponding geodesic flow is C^2, is uniquely ergodic.*

Note. If N is not orientable, then E^u will not be orientable. However, we still get a result similar to (3.6): namely, any W^u flow, corresponding to the generalized geodesic flow on the bundle of orthonormal 2-frames, is uniquely ergodic. (See [1, p. 10], [7].)

Finally, we describe the invariant measure for the W^u flow with our parametrization. Let μ be the measure which maximizes entropy for the Anosov flow. Then locally M looks like $B^u_\eta(x) \times B^{ws}_\eta(x)^\dagger$, and $\mu = \mu^u_x \times \mu^{ws}_x$, where μ^u_x is the measure μ_x on $B^u_\eta(x)$ that we used to construct our parametrization in Section 2, and μ^{ws}_x is invariant under the Poincaré return maps for the W^u flow (see [12, Section 4] and [15, p. 42]). From this it easy to see that μ is variant.

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† This is obtained by skewing the canonical coordinates product structure (1.2).

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